Westlake University
Number Theory, 2024-2025

Problem session

The problem set is composed of three parts and each part contains several questions. In parts I and II the questions are pairwise independent. In part III it is possible to use the statement of question (i) to solve question $(i+k)$.

Part I: Prove/Disprove

Prove or disprove the following statements (in the case of "if and only if" statements study both implications):

(1) Let k be a field; the polynomial ring $k[X,Y]$ is a Dedekind domain.

(2) Fix a $m, M \in \mathbb{Z}_{>0}$. The set

$$\{\alpha \in \overline{\mathbb{Q}} \colon [\mathbb{Q}(\alpha):\mathbb{Q}] \leq m,\ |\alpha| \leq M\}$$

is finite.

(3) Let K be a number field and let $x \in K$. Then $x \in \mathcal{O}_K^\times$ if and only if $N_{K|\mathbb{Q}}(x) = \pm 1$.

(4) Let K be a number field and let $x \in \mathcal{O}_K$. Then $x \in \mathcal{O}_K^\times$ if and only if $N_{K|\mathbb{Q}}(x) = \pm 1$.

(5) Let $\mathcal{O}$ be a Dedekind domain having only finitely many prime ideals, then $\mathcal{O}$ is a principal ideal domain.

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(6) Let K be a number field. There exists a finite extension $L \supseteq K$ with the following property: for any ideal $\mathfrak{a} \subset \mathcal{O}_K$ the extension $\mathfrak{a}\mathcal{O}_L$ is a principal ideal.

[Hint: the statement is true. Use the finiteness of the class number to prove it]

Part II: Some computations

(1) Let $K = \mathbb{Q}(\sqrt{-3})$. Show that $\mathcal{O}_K^\times \cong \mathbb{Z}/6\mathbb{Z}$.

Part III: Units and Pell's equation

Let $d \in \mathbb{Z}_{>0}$ squarefree with $d \not\equiv 1 \pmod 4$. Moreover let $K = \mathbb{Q}(\sqrt{d})$.

(1) Describe $\mathcal{O}_K$ and $\mathcal{O}_K^\times$.

(2) Show that there is exactly one <u>fundamental</u> unit $\epsilon = u + v\sqrt{d} \in \mathcal{O}_K^\times$ with $u, v \in \mathbb{Z}_{\geq 0}$. Moreover show that v is the smallest possible positive integer such that either $dv^2 + 1$ or $dv^2 - 1$ is a square.

(3) Describe the set $S_d = \{(x, y) \in \mathbb{Z}^2 \colon x^2 - dy^2 = 1\}$.

(4) Explain why in the case $d \equiv 1 \pmod 4$ the set S_d cannot be described with the same procedure.